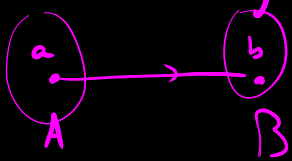


## Last Time: Relations

Recall: A relation  $A \xrightarrow{R} B$  is a subset  $R \subseteq A \times B$ .

NB: we built diagrams for relations write  $xRy$  for  $(x,y) \in R$ .



iff  $(a,b) \in R$ .

Special: relation on a set  $R \subseteq S \times S$ .

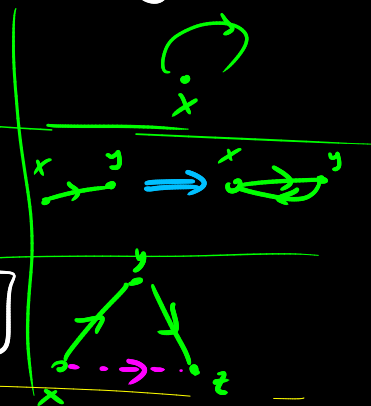
↳ use similar diagrams there (on one copy of  $S$ ).

Properties a relation on  $S$  might (or might not) satisfy:

① Reflexivity:  $\forall x \in S [xRx]$

② Symmetry:  $\forall x,y \in S [xRy \rightarrow yRx]$

③ Transitivity:  $\forall x,y,z \in S [xRy \wedge yRz \rightarrow xRz]$



Exercise: For each subset  $P \subseteq \{\text{reflexive, symmetric, transitive}\}$ , either build a relation on a set  $S$  which satisfies exactly the properties in  $P$ , or Prove no such relation exists.

↳ Ex: For  $P = \{\text{reflexive, transitive}\}$ , want:  
build a relation on some set  $S$  which is:  
① reflexive, ② transitive, ③ NOT symmetric.

Hints: ① Use the diagrams to visualize

② If an example exists, you need at most 4 elts in  $S$ .

Defn: An equivalence relation is a relation on a set  $S$  which is reflexive, symmetric, and transitive.

Ex: The relation depicted by  $\begin{matrix} \bullet & \xrightarrow{R} & \bullet \\ 1 & & 1 \end{matrix}$  is an equivalence relation. I.E.  $S = \{1\}$ ,  $R = \{(1,1)\}$ .

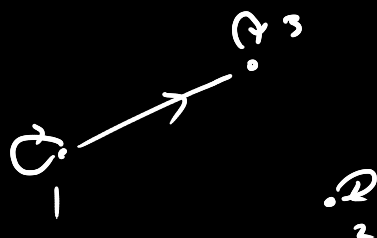
Reflexive: duh.

Symmetric: duh.

Transitive: duh.

> All trivial to verify...

Ex: The relation depicted below is NOT an equivalence relation.



Not symmetric

( $1 R 3$  but  $3 \not R 1$ )

Ex: Congruence modulo 5 is an equivalence relation on  $\mathbb{Z}$ .

pf: Recall  $x \equiv y \pmod{5}$  iff  $5 \mid (x-y)$ .  $\leftarrow$

Reflexive: Let  $x \in \mathbb{Z}$ . Now  $x-x=0=5 \cdot 0$ , and  $0 \in \mathbb{Z}$ , so  $5 \mid (x-x)$  yields  $x \equiv x \pmod{5}$ . Hence  $\equiv$  is reflexive.

Symmetric: Suppose  $x, y \in \mathbb{Z}$  satisfy  $x \equiv y \pmod{5}$ .

Thus  $5 \mid (x-y)$ , so  $x-y = 5k$  for some  $k \in \mathbb{Z}$  by definition of divisibility. Hence we have

$$y-x = -(x-y) = -5k = 5(-k), \text{ so using}$$

$-k \in \mathbb{Z}$  by closure of  $\mathbb{Z}$  under subtraction, we

see  $5 \mid (y-x)$  by definition of divisibility. Thus

$y \equiv x \pmod{5}$ ; hence  $\equiv$  is symmetric.

Transitive: Suppose  $x, y, z \in \mathbb{Z}$  satisfy  $x \equiv y \pmod{5}$  and

$y \equiv z \pmod{5}$ . Thus  $5 \mid (x-y)$  and  $5 \mid (y-z)$ , so

$x-y = 5k$  and  $y-z = 5m$  for some  $k, m \in \mathbb{Z}$

by definition of divisibility. Hence we compute

$$x - z = (x - y) + (y - z) = 5k + 5m = 5(k+m)$$

and  $k+m \in \mathbb{Z}$  by closure of  $\mathbb{Z}$  under addition,

so  $5|(x-z)$ , and thus  $x \equiv z \pmod{5}$ . Thus

$\equiv$  is transitive.

Hence,  $\equiv$  is an equivalence relation by definition (indeed, it is reflexive, symmetric, and transitive).  $\square$

Remark: If we replace each 5 above by  $n$ , we will have the bones of a proof that  $\equiv \pmod{n}$  is an equivalence relation on  $\mathbb{Z}$ .  $\smile$

NB: Equivalence relations are a useful mathematical tool. they crop up everywhere in maths and Computer Science...

## Functions

→ exactly one

Idea: A function is (roughly) an association table: it associates each input of a set  $S$  w/ a unique output.

Defn: A function from  $S$  to  $T$  is a relation  $S \xrightarrow{f} T$  such that every element of  $S$  is related to a unique element of  $T$ .

Ex: The relation below is a function:

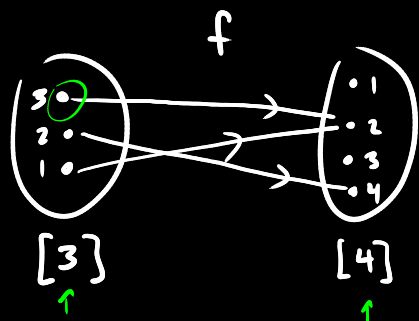


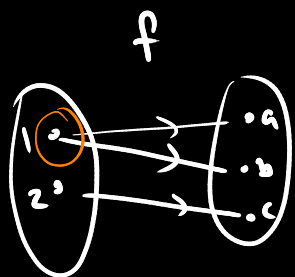
table:  
→

| x | f(x) |
|---|------|
| 1 | 2    |
| 2 | 4    |
| 3 | 2    |

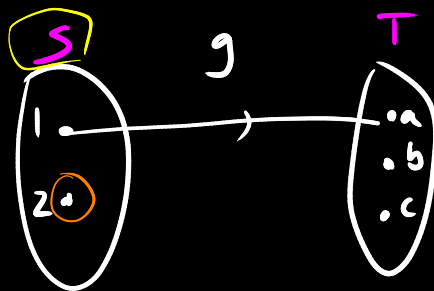
distinct inputs  
1, 3 have the  
same output 2.

(More Complete Definition): A function  $f: S \rightarrow T$  is a relation w/ every element of  $S$  relates to a unique element of  $T$ . The domain of  $f$  is  $\text{dom}(f) = S$  and the codomain of  $f$  is  $\text{cod}(f) = T$ . We say  $x$  maps to  $y$  when  $x f y$ . We also write  $y = f(x)$  when  $x f y$  (i.e.  $(x, y) \in f$ ).

Ex: The following are not functions:



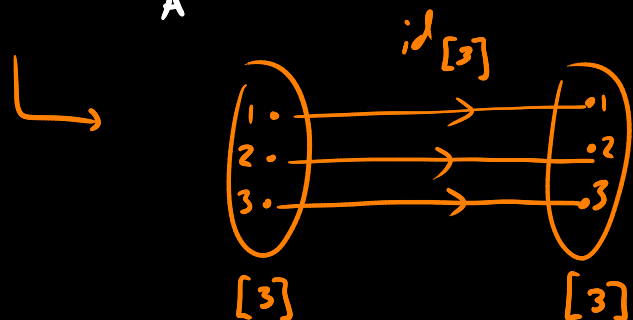
$1 f a$  and  $1 f b$



$2 \not g y$  for all  $y \in T$ .

Point: Every element of  $\text{dom}(f)$  has exactly one "arrow" leaving it, when  $f$  is a function.

Ex: Every set  $A$  has a special identity function.  
 $\text{id}_A: A \rightarrow A$  via  $\text{id}_A(a) = a$  for all  $a \in A$ .



i.e.  $\text{id}_{[3]} = \{(1,1), (2,2), (3,3)\}$   
as a relation on  $[3]$ .

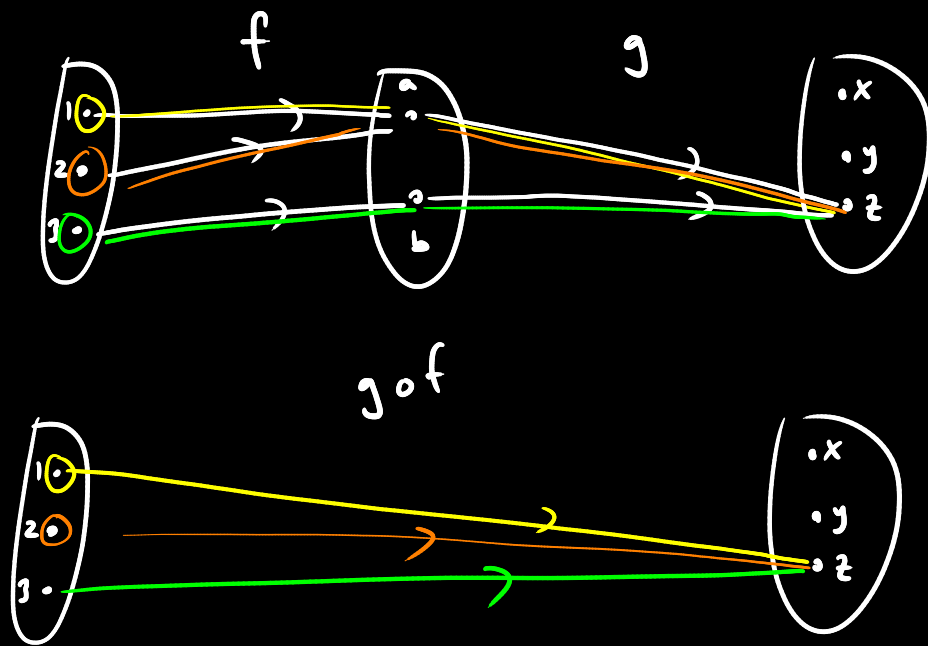
More generally,  $\text{id}_A = \{(a,a) : a \in A\}$  as a relation ...

Def<sup>n</sup>: Suppose  $f, g$  are functions w/  $\text{cod}(f) = \text{dom}(g)$ .

The composite of  $g$  with  $f$  is  $g \circ f: \text{dom}(f) \rightarrow \text{cod}(g)$  defined by  $(g \circ f)(x) = g(f(x))$ .

NB: This agrees with the definition from calculus...

Ex: Consider the functions  $f, g$  given below:



Ex: Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $g: \mathbb{Z} \rightarrow \mathbb{N}_0$  via

$f(x) = 3x + 7$  and  $g(x) = |x|$ . Then  
the composite  $g \circ f: \mathbb{Z} \rightarrow \mathbb{N}_0$

$$(g \circ f)(x) = g(f(x)) = g(3x + 7) = |3x + 7|.$$

On the other hand,  $f \circ f: \mathbb{Z} \rightarrow \mathbb{Z}$  satisfies

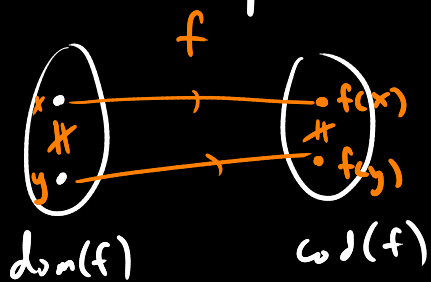
$$(f \circ f)(x) = f(f(x)) = f(3x + 7) = 3(3x + 7) + 7 = 9x + 28. \quad \square$$

Ex: The function  $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \quad \text{is con...} \quad \square$$

Defn: A function  $f$  is injective (aka into or one-to-one)  
when for all  $x, y \in \text{dom}(f)$  if  $f(x) = f(y)$ , then  $x = y$ .

NB: Another, less formal, way to say this is  
 "distinct inputs map to distinct outputs."



↑ the contrapositive: if  $x \neq y$ , then  $f(x) \neq f(y)$

Point:  $f$  injective:  $\forall x, y \in \text{dom}(f) [f(x) = f(y) \rightarrow x = y]$   
 $\equiv \forall x, y \in \text{dom}(f) [x \neq y \rightarrow f(x) \neq f(y)]$

Ex: is injective.

Ex: is NOT injective,  
 because  $f(1) = a = f(2)$ ,  
 but  $\underline{1 \neq 2}$ .

Ex: The function  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  w/  $f(x) = 3x + 7$  is injective. Let  $x, y \in \text{dom}(f) = \mathbb{Z}$  and suppose  $f(x) = f(y)$ . Hence  $3x + 7 = f(x) = f(y) = 3y + 7$ , which yields  $3x = 3y$  by subtracting 7 and thus  $x = y$  by cancelling 3's. Hence  $f(x) = f(y)$  implies  $x = y$ , so  $f$  is injective.  $\square$

NB:  $\neg (\forall x, y \in \text{dom}(f) [f(x) = f(y) \rightarrow x = y])$   
 $\equiv \exists x, y \in \text{dom}(f) (\neg [f(x) = f(y) \rightarrow x = y]) \equiv \exists x, y \in \text{dom}(f) [f(x) = f(y) \wedge x \neq y]$

Ex: The function  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  w/  $f(x) = 3x^2$  is not injective.  $f(1) = 3 \cdot 1^2 = 3 = 3 \cdot (-1)^2 = f(-1)$ , but  $1 \neq -1$ .  $\square$